

Quantum Capacity Approaching Codes for the Detected-Jump Channel

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The quantum channel capacity gives the ultimate limit for the rate at which quantum data can be reliably transmitted through a noisy quantum channel. Degradable quantum channels are among the few channels whose quantum capacities are known. Given the quantum capacity of a degradable channel, it remains challenging to find a practical coding scheme which approaches capacity. Here we discuss code designs for the detected-jump channel, a degradable channel with practical relevance describing the physics of spontaneous decay of atoms with detected photon emission. We show that this channel can be used to simulate a binary classical channel with both erasures and bit-flips. The capacity of the simulated classical channel gives a lower bound on the quantum capacity of the detected-jump channel. When the jump probability is small, it almost equals the quantum capacity. Hence using a classical capacity approaching code for the simulated classical channel yields a quantum code which approaches the quantum capacity of the detected-jump channel.

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Information theory was founded by Claude E. Shannon in 1948. In his landmark paper [1] he defined the notion of channel capacity, which is a tight upper bound on the amount of information that can be reliably transmitted over a noisy communication channel. The capacity of a channel is given by a single numerical quantity, characterizing the amount of information that can be transmitted asymptotically per channel use. Shannon also showed that it is possible to encode messages in such a way that the number of extra bits transmitted is as small as possible. Unfortunately his proof does not give any explicit recipe for these optimal codes. After decades of efforts the goal of finding explicit codes which reach the limits predicted by Shannon's original work has been achieved [2].

The origins of quantum information theory can be seen in the early 1990s. One of the questions addressed is the problem of reliable transmission of information through a quantum channel [3]. Unlike the situation for classical channels, one can define several capacities for quantum channels. The capacity of a quantum channel depends on the auxiliary resources allowed, the class of protocols allowed, and whether the information to be transmitted is classical or quantum [4, 5]. In this paper we discuss in particular the quantum capacity of a quantum channel which gives the ultimate bound on the rate at which quantum data can be reliably transmitted through a noisy quantum channel. This capacity is also called the one-way quantum capacity, where all communication is directly from the sender to the receiver over the noisy quantum channel.

Given a quantum channel Φ , the quantum capacity of

the channel is given by [4–6]

$$Q_C(\Phi) = \lim_{n \rightarrow \infty} \frac{1}{n} I^{\text{coh}}(\Phi^{\otimes n}), \quad (1)$$

where I^{coh} is the coherent information of a channel defined by

$$I^{\text{coh}}(\Phi) = \max_{\rho} (S(\Phi(\rho)) - S(\Phi^C(\rho))). \quad (2)$$

Here $S(\rho)$ is the von Neumann entropy of ρ , and Φ^C is the complementary channel of Φ .

Degradable quantum channels are among the few classes of channels whose quantum capacities are known. A channel Φ is degradable if there is another channel Ψ such that

$$\Psi \circ \Phi = \Phi^C. \quad (3)$$

It was shown in [7] that the capacity of a degradable channel satisfies

$$Q_C(\Phi) = I^{\text{coh}}(\Phi), \quad (4)$$

so that the capacity can be computed [8]. Given the quantum capacity of a degradable channel, the quantum coding theorem guarantees the existence of a scheme to encode the message such that the capacity can be achieved. However, it remains a challenge to find a practical coding scheme which approaches the capacity.

In this letter we discuss code designs for the detected-jump channel, which is a degradable channel with practical relevance describing the physics of spontaneous decay of atoms with detected photon emission. Our main

observation is that in the Hadamard basis, the detected-jump channel simulates a binary classical channel with both erasure and flip. This allows us to use classical codes to serve as the basis for a quantum code, resulting in good quantum jump codes with parameters better than those of any previously known codes. The capacity of the simulated classical channel also gives a lower bound for the quantum capacity of the detected-jump channel. When the jump probability is small, the scheme reaches almost the quantum capacity. Hence using classical codes approaching capacity for the simulated classical channel as the basis for the quantum codes results in quantum codes that approach the quantum capacity of the detected-jump channel.

Detected-Jump Channel The detected-jump channel was considered in [9]. It is a channel with practical relevance describing the physics of spontaneous decay of atoms with detected photon emission. The spontaneous decay is traditionally described by the jump channel (also called the amplitude damping channel, denoted by Φ_{AD}) described by the Kraus operators

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \quad (5)$$

that is,

$$\Phi_{AD}(\rho) = \sum_i A_i \rho A_i^\dagger, \quad (6)$$

If the photon emission of the spontaneous decay can be detected, then the corresponding channel, called the detected-jump channel (denoted by Φ_{DJ}) is given by

$$\Phi_{DJ}(\rho) = \sum_i \left(A_i \rho A_i^\dagger \right)_{\text{sys}} \otimes |i\rangle\langle i|_{\text{aux}}. \quad (7)$$

The complementary channel of Φ_{DJ} is given by

$$\Phi_{DJ}^C(\rho) = \sum_i \text{Tr}(A_i \rho A_i^\dagger) |i\rangle\langle i|_{\text{aux}}. \quad (8)$$

It is easy to see from (7) and (8) that Φ_{DJ}^C can be obtained from Φ_{DJ} by taking the trace of the first system, so Φ_{DJ} is degradable. Therefore, the quantum capacity of Φ_{DJ} can be directly calculated using Eq. (4).

Error-Correcting Codes for the Detected-Jump Channel The construction of quantum error-correcting codes for Φ_{DJ} has been discussed in [9–12]. Here we provide a new observation for the code construction. Starting from Eq. (7), applying a controlled-NOT operation from the auxiliary system to the system, we obtain an equivalent channel Φ'_{DJ} given by

$$\Phi'_{DJ}(\rho) = \sum_i \left(A'_i \rho A'_i{}^\dagger \right)_{\text{sys}} \otimes |i\rangle\langle i|_{\text{aux}}, \quad (9)$$

where

$$A'_0 = A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad \text{and} \quad A'_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{pmatrix}. \quad (10)$$

As both A'_0 and A'_1 are diagonal, the channel with these operators actually simulates some classical channel in the Hadamard basis. We clarify this fact by the following lemma.

Lemma 1 *In the Hadamard basis, Φ'_{DJ} simulates a binary classical channel Ξ with both erasure and bit flip, given by Fig. 1. The erasure probability is $p_E = \frac{\gamma}{2}$ and the bit flip probability is given by $p_F = \left[\frac{1}{2}(1 - \sqrt{1-\gamma}) \right]^2$.*

Proof: Note that $A'_1|+\rangle = \sqrt{\frac{\gamma}{2}}|1\rangle$ and $A'_1|-\rangle = -\sqrt{\frac{\gamma}{2}}|1\rangle$, i.e., in the Hadamard basis A'_1 erases all information, but at the same time this is indicated by the auxiliary state $|1\rangle_{\text{aux}}$. The probability for this event is $p_E = \frac{\gamma}{2}$. At the same time, in this basis A'_0 can be expressed as

$$A'_0 = \left[\frac{1}{2}(1 + \sqrt{1-\gamma}) \right] I + \left[\frac{1}{2}(1 - \sqrt{1-\gamma}) \right] Z \quad (11)$$

Hence, when no error was indicated by the auxiliary qubit and if we measure the system qubit in the Hadamard basis, the qubit will be flipped with probability $p_F = \left[\frac{1}{2}(1 - \sqrt{1-\gamma}) \right]^2$, or will not change with probability $1 - p_E - p_F$. \square

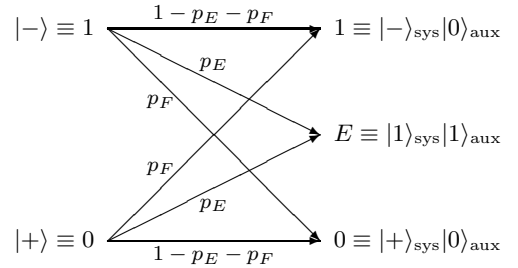


FIG. 1: A binary classical channel Ξ derived from the quantum channel Φ'_{DJ} by measuring the system in the basis $\{|+\rangle, |-\rangle\}$ and the auxiliary system in the basis $\{|0\rangle, |1\rangle\}$.

This lemma allows us to use classical codes for the classical channel Ξ in the Hadamard basis to serve as the basis for the quantum codes for the equivalent quantum channels Φ_{DJ} and Φ'_{DJ} .

In the following we construct a quantum error-correcting code \mathcal{Q} which is a subspace of $(\mathbb{C}^2)^{\otimes n}$, the space of n qubits. Recall that for a K -dimensional code space spanned by the orthonormal basis states $|\psi_i\rangle$, $i = 1, \dots, K$ and a set of errors \mathcal{E} there is a physical operation correcting all elements $E_\mu \in \mathcal{E}$ if the error correction conditions [13, 14] are satisfied:

$$\forall_{i,j,\mu,\nu} \quad \langle \psi_i | E_\mu^\dagger E_\nu | \psi_j \rangle = \alpha_{\mu\nu} \delta_{ij}, \quad (12)$$

where $\alpha_{\mu\nu}$ depends only on μ and ν . conditions (12) one can consider approximate error-correction [15], i.e. Eq. (12) is only fulfilled up to a certain order t .

Theorem 1 For an (n, K, d) classical code C , the residual error probability over the channel Ξ is of order $O(\gamma^d)$. From the classical code C , one obtains an $((n, K))$ quantum code which corrects errors of the quantum channel Φ_{DJ} up to order $O(\gamma^t)$ where $t = d - 1$.

Proof: An (n, K, d) classical code C can simultaneously correct e erasures and f errors provided $e + 2f < d$ (see, e.g., [16, Theorem 1.11.6]). Note that $p_E = \frac{\gamma}{2}$ and

$$p_F = \left[\frac{1}{2}(1 - \sqrt{1 - \gamma}) \right]^2 = \frac{1}{16}\gamma^2 + O(\gamma^3). \quad (13)$$

Therefore, using the code C , the residual error probability over the channel Ξ is improved to $O(\gamma^d)$.

Similarly, in order to correct errors of Φ_{DJ} up to order $O(\gamma^t)$, one needs to find an appropriate quantum code such that the error-correction conditions (12) hold up to order $O(\gamma^t)$ [15, 17, 18].

Based on the classical code C , we construct a corresponding quantum code \mathcal{Q} with basis

$$\{|\psi_{\mathbf{x}}\rangle = H^{\otimes n}|\mathbf{x}\rangle : \mathbf{x} \in C\}. \quad (14)$$

Now we prove that this code is sufficient for our task.

Setting $A = X + iY$ and $B = I - Z$, where X, Y, Z are Pauli operators for qubits, we have

$$A_1 = \frac{\sqrt{\gamma}}{2}A \quad \text{and} \quad A_0 = I - \frac{\gamma}{4}B + O(\gamma^2). \quad (15)$$

It can be shown that in order to improve the fidelity of the transmission through a detected-jump channel from $1 - \gamma$ to $1 - \gamma^{t+1}$, it is sufficient to satisfy the error-correction conditions Eq. (12) for e A -errors and f B -errors with $e + 2f \leq t$ [19, Section 8.7]. By the construction given in Eq. (14), this is equivalent to requiring that the classical code Ξ corrects e erasure errors and f bit flip errors (note here $d = t + 1$). \square

This theorem yields good quantum jump codes with parameters better than those previously known in [9–12].

Capacity Approaching Codes for the Detected-Jump Channel Contrary to the classical situation [2], there is almost nothing known about the construction of practical, quantum capacity approaching codes for quantum channels, not even for degradable channels. However, as a naive example consider a quantum channel Θ with Kraus operators $\sqrt{1-p}I$ and $\sqrt{p}X$. It is straightforward to calculate that the quantum capacity of this channel is given by $H(p)$, where H is the binary entropy. The corresponding classical channel is the binary symmetric channel with bit flip probability p , whose capacity is well-known to be $H(p)$, too. This is not a surprise. Although we use Θ to transmit quantum information, its behavior is exactly classical. Apparently, one can use capacity approaching codes for the binary symmetric channel as a basis for a quantum code for Θ , which then approaches the quantum capacity of Θ .

We borrow the idea from this naive example to design quantum capacity approaching codes for the detected-jump channel. Due to the previous discussion, we can use the classical codes for the simulated channel Ξ as a basis for a quantum code for the quantum channel Φ_{DJ} . The capacity of the classical channel Ξ gives a lower bound for the quantum capacity of the quantum channel $Q_C(\Phi_{DJ})$.

As Φ_{DJ} is degradable, its capacity can be computed using Eq. (4). It is given by

$$Q_C(\Phi_{DJ}) = \max_{x \in [0,1]} \{ (1 - \gamma x) \log(1 - \gamma x) - (1 - x) \log(1 - x) - (1 - \gamma)x \log(1 - \gamma)x \}. \quad (16)$$

Recall that for the transmission of classical information over a classical channel, the capacity is given by the maximal mutual information

$$\sup_{p_X} I(X; Y) \quad (17)$$

between the input X and the output Y , where the maximization is over all input distributions p_X [20]. The classical capacity of the simulated classical channel Ξ turns out to be

$$H(p_E, \frac{1 - p_E}{2}, \frac{1 - p_E}{2}) - H(p_E, p_F, 1 - p_E - p_F). \quad (18)$$

The capacities of the classical channel Ξ and the quantum channel Φ_{DJ} are plotted in Fig. 2.

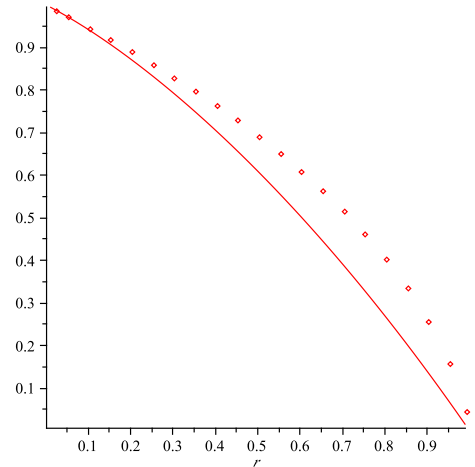


FIG. 2: The capacities for the region $\gamma \in [0, 1]$. The line is the capacity of the classical channel Ξ , and the dots give the capacity of the quantum channel Φ_{DJ} .

From Fig. 2 one can see that the capacity of the simulated classical channel gives a good lower bound on the quantum capacity of the detected-jump channel. When the jump probability is small, the lower bound almost equals the quantum capacity, see Fig. 3. Hence using a classical capacity approaching code for the simulated

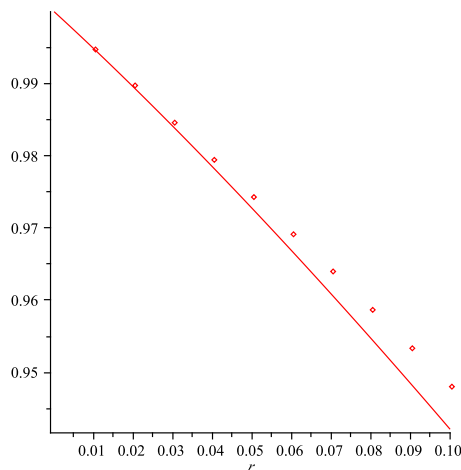


FIG. 3: The capacities for the region $\gamma \in [0, 0.1]$. The line is the capacity of the classical channel Ξ , and the dots give the capacity of the quantum channel Φ_{DJ} .

classical channel yields a quantum code which approaches the quantum capacity of the detected-jump channel.

Summary and Discussion We use classical codewords as basis for a quantum code to construct good quantum codes for the detected-jump channel with parameters better than those of any previously known codes. Our method gives a lower bound on the quantum capacity of the detected-jump channel. When the jump probability is small, it almost equals the quantum capacity. Hence using a classical capacity approaching code for the simulated classical channel yields a quantum code which approaches the quantum capacity of the detected-jump channel.

Our discussion is closely related to that of environment assisted channels [21, 22]. However, our detected-jump channel given in Eq. (7) is different from an environment assisted channel of the amplitude damping channel given in Eq. (5). Because we only allow the access of the environment in a fixed basis, the quantum capacity of Φ_{DJ} is lower than the environment assisted capacity of the amplitude damping channel.

Our result provides the first example of capacity-approaching codes of degradable quantum channels. We hope that our method sheds light on the problem of finding capacity-approaching codes for other degradable quantum channels.

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